

Hamiltonian extensions in quantum metrology

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Abstract

We study very generally to what extent the uncertainty with which a phase shift can be estimated in quantum metrology can be reduced by extending the Hamiltonian that generates the phase shift to an ancilla system with a Hilbert space of arbitrary dimension, and allowing arbitrary interactions between the original system and the ancilla. Such Hamiltonian extensions provide a general framework for open quantum systems, as well as for “non-linear metrology schemes” that have been investigated over the last few years. We prove that such Hamiltonian extensions cannot improve the sensitivity of the phase shift measurement when considering the quantum Fisher information optimized over input states.

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I. INTRODUCTION

Quantum metrology is concerned with the question of the ultimate precision with which certain parameters that characterize a physical system can be estimated based on measurements of the system. Such ultimate bounds arise from the quantum noise linked to the fundamental quantum mechanical nature of any physical systems. At the same time, there are situations where quantum mechanical effects such as entanglement or quantum interference can enhance the precision in certain parameter estimation schemes (see [1, 2] for recent reviews on such “quantum enhanced measurements”). The typical situation is the following: We are given a state $\rho(\theta)$ (or a collection of states) that depends on a parameter of interest θ . We suppose the form of the state completely known but not the value of θ which we want to estimate. “Estimate” rather than “measure” refers to the fact that θ may not correspond to any observable of the system, which implies that one first needs to measure some other observable, and then infer the value of θ from the measurement results. The tools of quantum parameter estimation theory provide different figures of merit to quantify with which precision we can estimate the parameter. Among these figures of merit, the Quantum Fisher Information (QFI) is known [3–6] to lead to an ultimate bound on the uncertainty of an unbiased estimator of θ (see (3) for a precise formulation).

From a physical point of view, it is worthwhile to consider the dynamics that imprints the parameter on the state. We thus start by an input state independent of the parameter, propagate it with a quantum channel \mathcal{E}_θ that depends on θ , and then look at the metrological properties of the output state. This is known as channel estimation. In such a framework, the object that we consider known and given is the channel \mathcal{E}_θ , and we have the freedom to still optimize over the input states.

It has been noticed early that in this channel estimation scheme, the use of entanglement can lead to an improvement in the precision of the estimation. By introducing an ancilla and entangling it with the initial probe *but still acting with the channel only on the initial probe*, *i.e.* using $\mathcal{E}_\theta \otimes \text{Id}$, an increase of the QFI can be observed for certain channels [7–9]. This is known as “channel extension” and we call a channel of the form $\mathcal{E}_\theta \otimes \text{Id}$ “extended channel”. The quantum channel \mathcal{E}_θ can be used in parallel protocols, sequentially, or as extended channel as described, but we still always use \mathcal{E}_θ to imprint the parameter. This is a natural point of view in quantum information as there the dynamics is described by

quantum channels.

A more physical point of view is that the fundamental physical object used to imprint the parameter on the state is not directly the channel but a given Hamiltonian $H(\theta)$. Obviously, to this Hamiltonian corresponds an evolution operator that gives rise to a unitary channel and we then go back to the channel estimation case. But when considering the concept of extensions we get a fundamental difference. Indeed, the natural way to extend an Hamiltonian is to introduce also an ancillary system, but then to add an Hamiltonian which allows interactions between both systems. We call such extensions “Hamiltonian extensions”, and the corresponding channels “Hamiltonian-extended channels”. These extensions describe a different situation than the one in channel extension, since there no interaction was allowed between the original system and the ancilla used for the extension.

In the present work we study in all generality the case of Hamiltonian extensions for a phase shift Hamiltonian of the form θG . The important question is whether such extensions can lead to an increased precision in the estimation process when optimizing over the input states. We show that this is not the case. Interestingly, in order to show this result for Hamiltonian extensions, we use a powerful theorem developed by Fujiwara and Imai on channel extensions, but only as a technical tool. The great generality of the situation described by Hamiltonian-extended phase shifts allows us to investigate some questions of quantum enhanced measurement. Notably, as the original phase shift may act already on a collection of subsystems, our bound can serve to investigate the effect of non-linear interactions [10–12]. The ancilla system may also be a heat-bath or a quantum bus, such that Hamiltonian extensions cover “decoherence-enhanced measurements” [13] or “coherent averaging” [14], too, as long the spectrum of all generators is bounded (see, however, the discussion of unbounded spectra in the Conclusions).

II. OPTIMAL CHANNEL ESTIMATION

A. Notation

Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ be the space of bounded linear operators acting on a Hilbert space \mathcal{H} of dimension d . A quantum channel \mathcal{E} is a *completely positive trace preserving* (CPTP) convex-linear map $\mathcal{E} : \mathcal{B} \rightarrow \mathcal{B}$ that maps a density matrix (*i.e.* a positive linear operator

with trace one) to another density matrix, $\rho \mapsto \sigma$. “Complete positivity” means that the channel should be a positive map (*i.e.* maps positive operators to positive ones), but also that the extension $\mathcal{E} \otimes \text{Id}$ of the channel to ancillary Hilbert spaces $\tilde{\mathcal{H}}$, where it acts by the identity operator, should be a positive map, *i.e.* $(\mathcal{E} \otimes \text{Id})(A) \geq 0$ for any positive operator A in $\mathcal{B}(\mathcal{H} \otimes \tilde{\mathcal{H}})$, the space of bounded operator acting on the bipartite system $\mathcal{H} \otimes \tilde{\mathcal{H}}$ [15]. Trace preservation is defined as $\text{tr}[\mathcal{E}(\rho)] = \text{tr}[\rho]$, and convex linearity as $\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i)$ for all p_i with $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. According to Kraus’ theorem, a quantum channel can be represented as

$$\mathcal{E}(\rho) = \sum_{i=1}^q A_i \rho A_i^\dagger, \quad (1)$$

where the set of q Kraus operators $\mathcal{A} = \{A_i\}_{i=1,\dots,q}$ is called a q -Kraus decomposition of \mathcal{E} , and $\sum_{i=1}^q A_i^\dagger A_i = \mathcal{I}$, the identity operator on the Hilbert space \mathcal{H} [16]. The Kraus representation (1) is not unique: Giving a reference q -Kraus decomposition $\mathcal{A}(\theta) = \{A_j(\theta)\}_{j=1,\dots,q}$ of a channel \mathcal{E}_θ , we can construct all the other q -Kraus decompositions through the unitary matrices of size q ,

$$\left\{ B_j(\theta) = \sum_k u_{jk}(\theta) A_k(\theta) \right\}_{j=1,\dots,q}, \quad (2)$$

with $u_{ij}(\theta) = (U(\theta))_{ij}$ a unitary matrix. The set of all q -Kraus decompositions of a channel is called the q -Kraus ensemble and is noted \mathcal{A}_q . The smallest possible number q of Kraus operators is known as “Kraus rank”. It can be obtained as the number of non-vanishing eigenvalues of the Choi-matrix of the channel (see [17]).

B. Quantum Fisher Information

Quantum parameter estimation theory (q-pet) provides a powerful tool for calculating the smallest uncertainty achievable when estimating a parameter θ encoded in a state $\rho(\theta)$. Central object in the theory is the quantum Fisher information which enters in the quantum Cramér-Rao bound (QCRB). We first review QFI for a state and then consider channel estimation.

1. QFI for a quantum state

The QCRB provides a lower bound on the variance of an unbiased estimator $\hat{\theta}_{\text{est}}$ of θ . Its importance arises from the facts that it is optimized already over all possible POVM measurements (measurements that include and generalize projective von Neumann measurements to account for quantum-probes to which the quantum system is coupled and which are then measured via projective von-Neumann measurements [18]), and all possible data analysis schemes in the form of unbiased estimators. These are estimators that on the average give back the true value of the parameter. The QCRB is given by

$$\text{Var}(\hat{\theta}_{\text{est}}) \geq \frac{1}{M I(\rho(\theta); \theta)}, \quad (3)$$

with M the number of independent measurements and $I(\rho(\theta); \theta)$ the quantum Fisher information (QFI). The QCRB is reachable asymptotically in the limit of an infinite number of measurements.

The QFI is given by $I(\rho(\theta); \theta) = \text{tr}[L_\theta^2 \rho(\theta)]$, where the symmetric logarithmic derivative L_θ is defined implicitly by $2 \text{d}\rho(\theta)/\text{d}\theta = L_\theta \rho(\theta) + \rho(\theta) L_\theta$. In [5] it was shown that $I(\rho(\theta); \theta)$ is linked to the distance between the two infinitesimally closed states $\rho(\theta)$ and $\rho(\theta + d\theta)$. More specifically we have $\lim_{d\theta \rightarrow 0} d_B(\rho(\theta), \rho(\theta + d\theta))^2 / d\theta^2 = I(\rho(\theta); \theta)/4$, where the Bures distance d_B between two states σ and τ is defined as $d_B(\sigma, \tau) = (2 - 2 \text{tr}[(\sqrt{\tau} \sigma \sqrt{\tau})^{1/2}])^{1/2}$. The QCRB thus offers the physically intuitive picture that the parameter θ can be measured the more precisely the more strongly the state $\rho(\theta)$ depends on it.

The QFI enjoys some very useful properties. First it is monotonous under θ -independent channels \mathcal{E}

$$I(\mathcal{E}(\rho(\theta)); \theta) \leq I(\rho(\theta); \theta), \quad (4)$$

with equality for θ -independent unitary channels \mathcal{U} [19] (see eq.(15) for the definition of a unitary channel). The QFI is also convex, meaning that for two density matrices $\rho(\theta)$ and $\sigma(\theta)$ and $0 \leq \lambda \leq 1$ we have [7]

$$I(\lambda \rho(\theta) + (1 - \lambda) \sigma(\theta); \theta) \leq \lambda I(\rho(\theta); \theta) + (1 - \lambda) I(\sigma(\theta); \theta). \quad (5)$$

Finally, the QFI is additive, in the sense that

$$I(\rho(\theta) \otimes \sigma(\theta); \theta) = I(\rho(\theta); \theta) + I(\sigma(\theta); \theta). \quad (6)$$

2. Channel QFI

When we want to know how precisely the parameter characterizing a quantum channel can be estimated, we have the additional freedom of optimizing over the input state. We define the *channel quantum Fisher information* $C(\mathcal{E}_\theta; \theta)$ of a channel \mathcal{E}_θ as

$$C(\mathcal{E}_\theta; \theta) = \max_{\rho \in \mathcal{B}(\mathcal{H})} I(\mathcal{E}_\theta(\rho); \theta). \quad (7)$$

Due to the convexity of the QFI, it is enough to maximize over the pure states,

$$\max_{\rho \in \mathcal{B}(\mathcal{H})} I(\mathcal{E}_\theta(\rho); \theta) = \max_{|\psi\rangle \in \mathcal{H}} I(\mathcal{E}_\theta(|\psi\rangle\langle\psi|); \theta). \quad (8)$$

3. Extensions of a quantum channel

In this paper we are interested in the Hamiltonian extension, which differs from channel extension. Nevertheless, channel extensions are needed in our calculation as a technical tool, and we thus start by presenting how this works.

As quantum channels are *completely* positive trace preserving maps it is natural to consider extensions of channels as

$$\mathcal{E}_\theta \rightarrow \mathcal{E}_\theta \otimes \mathcal{A}, \quad (9)$$

where \mathcal{A} is an arbitrary channel acting on an ancilla system. Extensions can be written as a concatenation,

$$\mathcal{E}_\theta \otimes \mathcal{A} = (\mathcal{E}_\theta \otimes \text{Id}) \circ (\text{Id} \otimes \mathcal{A}) = (\text{Id} \otimes \mathcal{A}) \circ (\mathcal{E}_\theta \otimes \text{Id}). \quad (10)$$

Using the monotonicity of the QFI we have

$$I((\mathcal{E}_\theta \otimes \mathcal{A})(\rho); \theta) = I((\text{Id} \otimes \mathcal{A}) \circ (\mathcal{E}_\theta \otimes \text{Id})(\rho); \theta) \leq I((\mathcal{E}_\theta \otimes \text{Id})(\rho); \theta) \quad (11)$$

and the equality is achieved when $(\text{Id} \otimes \mathcal{A})$ is a unitary channel, *i.e.* when \mathcal{A} is a unitary channel [7]. Hence, in terms of estimation, it is enough to consider extensions by the identity, and with “*channel extension*”, we will always refer to *extension by the identity*. The situation is depicted in figure 1. In certain cases it was noticed that this allows a better estimation of the parameter, although we act with the identity on the ancillary Hilbert space [7–9].

Fujiwara and Imai provided a theorem to calculate the channel QFI of an extended channel in an efficient way:

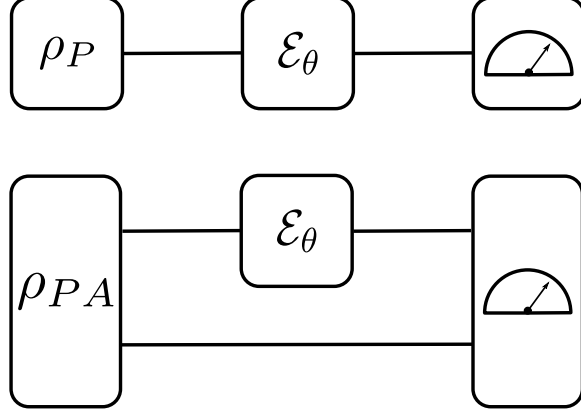


FIG. 1. Channel extension for an arbitrary channel \mathcal{E}_θ . Top scheme: original channel of the probe (P). Bottom scheme: channel extension of \mathcal{E}_θ to an ancilla (A) on which one acts with the identity operation.

Theorem 1 (Channel QFI of extended channels [20]). *For a one parameter family of quantum channels $\{\mathcal{E}_\theta\}$ and for any natural number q such that $q \geq \text{rank}(\mathcal{E}_\theta)$, we have*

$$C(\mathcal{E}_\theta \otimes \text{Id}; \theta) = 4 \min_{\mathcal{A}(\theta) \in \mathcal{A}_q} \left\| \sum_{j=1}^q \dot{A}_j^\dagger(\theta) \dot{A}_j(\theta) \right\|_\infty, \quad (12)$$

with $\mathcal{A}(\theta) = \{A_j(\theta)\}_{j=1, \dots, q}$, $\dot{A}_j(\theta) = dA_j(\theta)/d\theta$, and where $\|\bullet\|_\infty$ is the infinity norm of \mathcal{H} .

The infinity norm $\|\bullet\|_\infty$ is also known as the operator norm, or the spectral norm. It is defined as $\|A\|_\infty = \max\{\|Au\| : u \in \mathcal{H}, \|u\| = 1\}$ where $\|\bullet\|$ corresponds to the usual Euclidean norm in \mathcal{H} . The infinity norm obeys the submultiplicativity property

$$\|XY\|_\infty \leq \|X\|_\infty \|Y\|_\infty, \quad (13)$$

and also

$$\|X^\dagger X\|_\infty = \|X\|_\infty^2. \quad (14)$$

III. HAMILTONIAN EXTENSIONS

We now come to the core of this paper, namely the concept of Hamiltonian extensions. Since in this framework we want to describe the dynamics of a system by Hamiltonians, the corresponding channels are unitary channels. A unitary channel \mathcal{U}_H is defined as

$$\mathcal{U}_H(\rho) = U_H \rho U_H^\dagger, \quad (15)$$

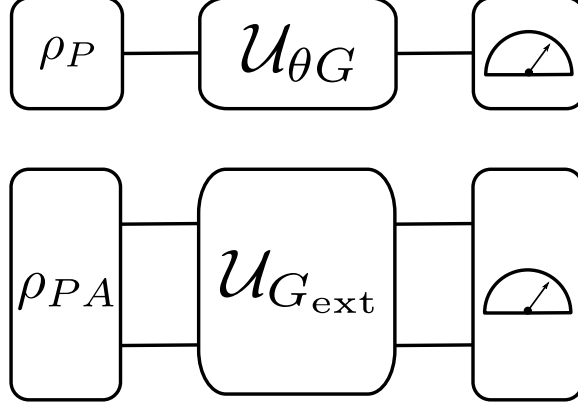


FIG. 2. Hamiltonian extension for a phase shift channel $\mathcal{U}_{\theta G}$. Top scheme: original phase shift channel. Bottom scheme: Hamiltonian-extended phase-shift channel (see eq.(17)).

with $U_H = e^{-iH(\theta)}$ a unitary matrix parametrized by θ . The specific case of phase shift channels $\mathcal{U}_{\theta G}(\rho)$ is given by

$$\mathcal{U}_{\theta G}(\rho) = U_{\theta G} \rho U_{\theta G}^\dagger, \quad (16)$$

with $U_{\theta G} = e^{-i\theta G}$ where G is the generator of the phase shift. Throughout this paper we will consider only generators that have a bounded spectrum.

To extend this Hamiltonian we first introduce an ancillary system with a Hilbert space of arbitrary dimension d' , and then add an arbitrary new Hamiltonian H_{int} that acts on both subsystems. We thus get our *Hamiltonian-extended phase shift* (see figure 2)

$$G_{\text{ext}}(\theta) = \theta G \otimes \mathcal{I} + H_{\text{int}}. \quad (17)$$

Notice that in particular H_{int} can contain also a part that acts on the second subsystem alone, *i.e.* in the language of open quantum systems, one can identify H_{int} in eq.(17) with the sum of the usual interaction Hamiltonian H_{int} and the Hamiltonian of the environment H_{env} . Importantly, the channel corresponding to the Hamiltonian-extended phase shift does not correspond trivially to a channel extension, as there one acts only with the identity operator on the ancillary system.

The important question is whether the unitary channel corresponding to the Hamiltonian-extended phase shift $\mathcal{U}_{G_{\text{ext}}}$ can have a greater channel QFI than the original phase shift channel QFI. *I.e.* we have to compare $C(\mathcal{U}_{G_{\text{ext}}}; \theta)$ and $C(\mathcal{U}_{\theta G}; \theta)$. The answer is given by the following theorem, which is the main result of this paper:

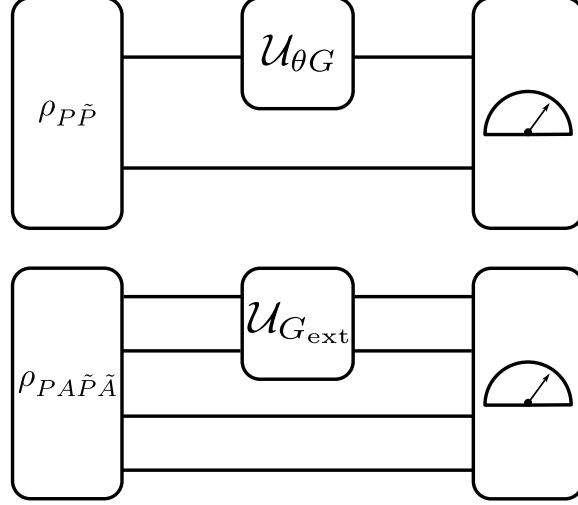


FIG. 3. Technical channel extension of a Hamiltonian-extended phase shift channel. We use channel extension for phase shift and Hamiltonian-extended phase shift in order to calculate the channel QFI with the help of the theorem 1 from Fujiwara and Imai [20]. The subscripts P and A correspond to the "physical" probe and ancilla, while \tilde{P} and \tilde{A} refer to ancillary systems used for the technical channel extension.

Theorem 2 (Channel QFI for Hamiltonian-extended phase shift). *Let $\mathcal{U}_{\theta G}$ be a phase shift channel and $\mathcal{U}_{G_{\text{ext}}}$ the corresponding Hamiltonian-extended channel (eq.(17)). Then the channel QFI of the Hamiltonian-extended phase shift channel is bounded by the channel QFI of the original phase shift channel:*

$$C(\mathcal{U}_{G_{\text{ext}}}; \theta) \leq C(\mathcal{U}_{\theta G}; \theta) . \quad (18)$$

A. Channel QFI of a phase shift

To prove this theorem we will make technical use of channel extensions of both the original phase shift channel and the Hamiltonian-extended phase shift channel. The situation is represented in figure 3. We first show the following lemma:

Lemma 1 (Invariance of the channel QFI of phase shift channels under channel extension). *Consider a phase shift channel $\mathcal{U}_{\theta G}$ with generator G . The channel QFI of the extended channel $\mathcal{U}_{\theta G} \otimes \text{Id}$ is equal to the original channel QFI:*

$$C(\mathcal{U}_{\theta G} \otimes \text{Id}; \theta) = C(\mathcal{U}_{\theta G}; \theta) . \quad (19)$$

This shows that phase shift channels do not benefit in terms of channel QFI from channel extensions.

Proof. The lemma follows by comparing the channel QFI of the extended phase shift channel and of the original phase shift channel. In both cases the QFI is maximized by a pure state, and we also know that the QFI for a phase shift channel for a pure state is equal to four times the variance of the generator. Thus we have

$$C(\mathcal{U}_{\theta G}; \theta) = 4 \max_{|\psi\rangle \in \mathcal{H}} \text{Var}[G, |\psi\rangle\langle\psi|] , \quad (20)$$

$$C(\mathcal{U}_{\theta G} \otimes \text{Id}; \theta) = 4 \max_{|\varphi\rangle \in \mathcal{H} \otimes \mathcal{H}} \text{Var}[G \otimes \mathcal{I}, |\varphi\rangle\langle\varphi|] . \quad (21)$$

To see that these two quantities are equal, it is enough to consider the smallest and largest eigenvalues of G , g_1 and g_d , respectively. The corresponding eigenvectors are noted $|\psi_1\rangle$ and $|\psi_d\rangle$. Popoviciu's inequality [21] states that the variance of a random variable X with lower and upper bound $a = \inf(X)$ and $b = \sup(X)$ respectively, is upper bounded by $|b - a|^2 / 4$. Since extending the Hamiltonian by the identity does not change the value of the eigenvalues but just their multiplicity, this implies that both variances are upper bounded by $|g_1 - g_d|^2 / 4$,

$$\begin{aligned} \max_{|\psi\rangle \in \mathcal{H}} \text{Var}[G, |\psi\rangle\langle\psi|] &\leq |g_1 - g_d|^2 / 4 , \\ \max_{|\varphi\rangle \in \mathcal{H} \otimes \mathcal{H}} \text{Var}[G \otimes \mathcal{I}, |\varphi\rangle\langle\varphi|] &\leq |g_1 - g_d|^2 / 4 , \end{aligned}$$

The proof is completed by noticing that both bounds are saturated, respectively, by the state $|\psi_{\text{opt}}\rangle = (|\psi_1\rangle + |\psi_d\rangle)/\sqrt{2}$ and $|\varphi_{\text{opt}}\rangle = |\psi_{\text{opt}}\rangle \otimes |\tilde{\varphi}\rangle$ with $|\tilde{\varphi}\rangle$ an arbitrary state. \square

B. Channel QFI of a general unitary channel

We now go back to the general case with the Hamiltonian $H(\theta)$. We have the following proposition:

Proposition 1 (Channel QFI of a general unitary channel). *Consider a general unitary channel \mathcal{U}_H with Hamiltonian $H(\theta)$. The channel QFI of the extended channel $\mathcal{U}_H \otimes \text{Id}$ is written*

$$C(\mathcal{U}_H \otimes \text{Id}; \theta) = 4 \min_{x \in \mathbf{R}} \|\dot{U}_H - i x U_H\|_{\infty}^2 . \quad (22)$$

Proof. The proof is a direct application of theorem 1 by Fujiwara and Imai. We start by taking a reference Kraus operator (we work with $q = 1$) U_H . The 1-Kraus ensemble \mathcal{A}_1 is generated by the reference Kraus operator as $\mathcal{A}_1 = \{A_1(\theta) = e^{-i x(\theta)} U_H\}$. The derivative of the elements of the 1-Kraus ensemble gives $\dot{A}_1(\theta) = e^{-i x(\theta)} (\dot{U}_H - i \dot{x}(\theta) U_H)$. Using property (14) of the infinity norm we obtain the desired result with $x \equiv \dot{x}(\theta)$. [22] \square

C. Linear shift and centered Hamiltonians

Proposition 2 (Linear shift of generators). *Consider a linear shift proportional to θ for a general unitary evolution \mathcal{U}_H generated by $H(\theta)$,*

$$H_\alpha(\theta) = H(\theta) + \theta \alpha \mathcal{I} , \quad (23)$$

and define the channel \mathcal{U}_{H_α} by

$$\mathcal{U}_{H_\alpha}(\rho) = U_{H_\alpha} \rho U_{H_\alpha}^\dagger , \quad (24)$$

with $U_{H_\alpha} = e^{-i H_\alpha(\theta)}$. Then the channel QFI is invariant under such linear shifts

$$C(\mathcal{U}_{H_\alpha} ; \theta) = C(\mathcal{U}_H ; \theta) . \quad (25)$$

Proof. We can expand U_{H_α} as

$$U_{H_\alpha} = e^{-i(H(\theta) + \theta \alpha \mathcal{I})} = e^{-i H(\theta)} e^{-i \theta \alpha \mathcal{I}} = e^{-i \theta \alpha} e^{-i H(\theta)} = e^{-i \theta \alpha} U_H . \quad (26)$$

When applying this channel to a state ρ we get

$$U_{H_\alpha} \rho U_{H_\alpha}^\dagger = e^{-i \theta \alpha} U_H \rho (e^{-i \theta \alpha} U_H)^\dagger = U_H \rho U_H^\dagger . \quad (27)$$

Both channels produce the same state, since the shift just adds a global phase factor. Therefore the channel QFI for both channels are equal,

$$C(\mathcal{U}_{H_\alpha} ; \theta) = C(\mathcal{U}_H ; \theta) . \quad (28)$$

In the same fashion we obtain for extended unitary channels

$$C(\mathcal{U}_{H_\alpha} \otimes \text{Id} ; \theta) = C(\mathcal{U}_H \otimes \text{Id} ; \theta) . \quad (29)$$

\square

We now go back to the case of unitary evolution in the form of phase shifts with a Hamiltonian $H(\theta) = \theta G$. We say that a generator is centered and use the notation \tilde{G} if and only if its largest and smallest eigenvalues obey $\tilde{g}_1 = -\tilde{g}_d$. We then have the following proposition:

Proposition 3 (Centered phase shift). *The channel QFI of the extended centered phase shift channel is equal to*

$$C(\mathcal{U}_{\theta\tilde{G}} \otimes \text{Id}; \theta) = (\tilde{g}_1 - \tilde{g}_d)^2 = 4\tilde{g}_1^2 = 4\|\tilde{G}\|_\infty^2. \quad (30)$$

Proof. The proof is direct when making use of the fact that the infinity norm of a Hermitian operator is given by the largest absolute value of its eigenvalues. Since the Hamiltonian is centered, both its extremal eigenvalues have the same absolute value $|\tilde{g}_1| = |\tilde{g}_d|$ which gives the desired result. \square

With eqs.(30,19) we obtain for centered phase shift channels

$$C(\mathcal{U}_{\theta\tilde{G}}; \theta) = 4\|\tilde{G}\|_\infty^2. \quad (31)$$

D. Extensions of phase shift Hamiltonians

The extended phase shift (17) corresponds to a general unitary channel with Hamiltonian $G_{\text{ext}}(\theta)$ and thus the results of section III B hold. In particular we have from eq.(22)

$$C(\mathcal{U}_{G_{\text{ext}}} \otimes \mathcal{I}; \theta) = 4 \min_{x \in \mathbf{R}} \|\dot{U}_{G_{\text{ext}}} - i x U_{G_{\text{ext}}}\|_\infty^2, \quad (32)$$

with $U_{G_{\text{ext}}} = e^{-i G_{\text{ext}}(\theta)} = e^{-i(\theta G \otimes \mathcal{I} + H_{\text{int}})}$. In the following we will find an upper bound to the right hand side of equation (32).

Lemma 2 (Upper bound for $C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta)$). *The channel QFI $C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta)$ is upper bounded by four times the norm of the original generator of the phase shift:*

$$C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta) \leq 4\|G\|_\infty^2. \quad (33)$$

Proof. Since the norm is positive, the minimum of its square equals the square of its minimum, and we obtain

$$C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta) = 4 \left(\min_{x \in \mathbf{R}} \|\dot{U}_{G_{\text{ext}}} - i x U_{G_{\text{ext}}}\|_\infty \right)^2. \quad (34)$$

Using the triangle inequality, we have

$$\|\dot{U}_{G_{\text{ext}}} - ixU_{G_{\text{ext}}}\|_{\infty} \leq \|\dot{U}_{G_{\text{ext}}}\|_{\infty} + |x| \|U_{G_{\text{ext}}}\|_{\infty} . \quad (35)$$

Minimizing over x gives

$$\min_{x \in \mathbf{R}} \left(\|\dot{U}_{G_{\text{ext}}}\|_{\infty} + |x| \|U_{G_{\text{ext}}}\|_{\infty} \right) = \|\dot{U}_{G_{\text{ext}}}\|_{\infty} , \quad (36)$$

which is reached for $x = 0$ since the three terms $\|\dot{U}_{G_{\text{ext}}}\|_{\infty}$, $|x|$ and $\|U_{G_{\text{ext}}}\|_{\infty}$ are all positive.

We are thus left with

$$C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta) \leq 4 \|\dot{U}_{G_{\text{ext}}}\|_{\infty}^2 . \quad (37)$$

Now we try to find an upper bound for $\|\dot{U}_{G_{\text{ext}}}\|_{\infty}$. To do so we use the Trotter unitary product formula for a pair of Hermitian operators A and B and their sum $C = A + B$ which states that

$$(e^{-itA/N} e^{-itB/N})^N - e^{-itC} \rightarrow 0 , \quad N \rightarrow \infty , \quad (38)$$

with a uniform convergence for $t \in \mathbf{R}$ [23, 24]. By setting $A = \theta G \otimes \mathcal{I}$, $B = H_{\text{int}}$ and $t = 1$ we get

$$U_{G_{\text{ext}}} = \lim_{N \rightarrow \infty} (e^{-i\theta G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N . \quad (39)$$

We need to calculate the derivative of this operator. For this we will make use of the following theorem to interchange the orders of the limits.

Theorem 3 (Interchange of orders of limits [25]). *Let E be a topological space, F a metric space, A a subset of E and f_0, f_1, \dots, f_n a sequence of maps from A in F uniformly converging to f . Let also a be an adherent point [26] of A .*

If, for each n , $f_n(x)$ has a limit when $x \rightarrow a$ through a sequence of values in A , and if F is complete, then $f(x)$ has a limit when $x \rightarrow a$, and furthermore

$$\lim_{\substack{x \rightarrow a \\ x \in A}} f(x) = \lim_{n \rightarrow \infty} \left(\lim_{\substack{x \rightarrow a \\ x \in A}} f_n(x) \right) . \quad (40)$$

In order to use this theorem we write the derivative of the operator as

$$\dot{U}_{G_{\text{ext}}} = \frac{d}{d\theta} U_{G_{\text{ext}}} = \lim_{\varepsilon \rightarrow 0} \frac{U_{G_{\text{ext}}} |_{\theta+\varepsilon} - U_{G_{\text{ext}}} |_{\theta}}{\varepsilon} . \quad (41)$$

Using theorem 3, we have

$$\dot{U}_{G_{\text{ext}}} = \lim_{\varepsilon \rightarrow 0} \frac{\lim_{N \rightarrow \infty} (e^{-i(\theta+\varepsilon)G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N - \lim_{N \rightarrow \infty} (e^{-i\theta G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N}{\varepsilon} \quad (42)$$

$$= \lim_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{(e^{-i(\theta+\varepsilon)G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N - (e^{-i\theta G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N}{\varepsilon} \quad (43)$$

$$= \lim_{N \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{(e^{-i(\theta+\varepsilon)G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N - (e^{-i\theta G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N}{\varepsilon} \quad (44)$$

$$= \lim_{N \rightarrow \infty} \frac{d}{d\theta} (e^{-i\theta G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^N. \quad (45)$$

Evaluating the derivative, we obtain

$$\dot{U}_{G_{\text{ext}}} = \lim_{N \rightarrow \infty} \sum_{i=1}^N (e^{-i\theta G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^{i-1} \times (-i G \otimes \mathcal{I}/N) (e^{-i\theta G \otimes \mathcal{I}/N} e^{-iH_{\text{int}}/N})^{N-(i-1)}. \quad (46)$$

Using the submultiplicativity property (13) of the infinity norm among with the triangle inequality, we have for an arbitrary set of operator $\{A_{i,j}\}$

$$\left\| \sum_i \prod_j A_{i,j} \right\|_{\infty} \leq \sum_i \prod_j \|A_{i,j}\|_{\infty}. \quad (47)$$

Then, using (47) and the fact that limit and norm commute,

$$\left\| \lim_{N \rightarrow \infty} A_N \right\|_{\infty} = \lim_{N \rightarrow \infty} \|A_N\|_{\infty}, \quad (48)$$

we obtain

$$\begin{aligned} \|\dot{U}_{G_{\text{ext}}}\|_{\infty} &\leq \lim_{N \rightarrow \infty} \sum_{i=1}^N (\|e^{-i\theta G \otimes \mathcal{I}/N}\|_{\infty} \|e^{-iH_{\text{int}}/N}\|_{\infty})^{i-1} \|-i G \otimes \mathcal{I}/N\|_{\infty} \\ &\quad \times (\|e^{-i\theta G \otimes \mathcal{I}/N}\|_{\infty} \|e^{-iH_{\text{int}}/N}\|_{\infty})^{N-(i-1)}. \end{aligned} \quad (49)$$

Since the unitary operators have norm one, the result simplifies to

$$\|\dot{U}_{G_{\text{ext}}}\|_{\infty} \leq \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{N} \|G \otimes \mathcal{I}\|_{\infty} = \|G \otimes \mathcal{I}\|_{\infty} = \|G\|_{\infty}. \quad (50)$$

Finally we are left with

$$C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta) \leq 4\|G\|_{\infty}^2.$$

□

E. Centering extended Hamiltonians

We now combine the result on centered phase shift and the conservation of the channel QFI over a θ -linear shift of the generator and apply them to Hamiltonian-extended phase shift. We consider the θ -linear shifted Hamiltonian-extended phase shift

$$G_{\text{ext},\alpha}(\theta) = \theta(G \otimes \mathcal{I} + \alpha \mathcal{I} \otimes \mathcal{I}) + H_{\text{int}} = \theta G_{\alpha} \otimes \mathcal{I} + H_{\text{int}} , \quad (51)$$

with the shifted phase shift generator

$$G_{\alpha} = G + \alpha \mathcal{I} . \quad (52)$$

By choosing $\alpha = \alpha_c = \frac{g_1 + g_d}{2}$ we obtain a centered generator $\tilde{G}_{\alpha_c} \equiv G_{\alpha_c}$ (and the corresponding Hamiltonian-extended centered phase shift G_{ext,α_c}), with extremal eigenvalues $\tilde{g}_1 = g_1 - \frac{g_1 + g_d}{2} = \frac{g_1 - g_d}{2}$ and $\tilde{g}_d = g_d - \frac{g_1 + g_d}{2} = \frac{g_d - g_1}{2}$. We thus have $\tilde{g}_1 = -\tilde{g}_d$ showing that the generator is indeed centered.

F. Proof of main theorem

We have now all ingredients to prove theorem 2.

Proof. We start by the channel QFI of the extended phase shift channel $C(\mathcal{U}_{G_{\text{ext}}}; \theta)$. This quantity is bounded by its channel extension,

$$C(\mathcal{U}_{G_{\text{ext}}}; \theta) \leq C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta) .$$

Using the fact that the channel QFI of the extended channel is invariant under a θ -linear shift, eq.(29), we have

$$C(\mathcal{U}_{G_{\text{ext}}} \otimes \text{Id}; \theta) = C(\mathcal{U}_{G_{\text{ext},\alpha_c}} \otimes \text{Id}; \theta) ,$$

where α_c is chosen such that \tilde{G}_{α_c} is a centered generator.

Because $\mathcal{U}_{G_{\text{ext},\alpha_c}}$ is a Hamiltonian-extended phase shift channel, we know that the channel QFI of its extension is bounded by the norm of the corresponding generator, eq.(33),

$$C(\mathcal{U}_{G_{\text{ext},\alpha_c}} \otimes \text{Id}; \theta) \leq 4 \|\tilde{G}_{\alpha_c}\|_{\infty}^2 .$$

Since \tilde{G}_{α_c} is a centered generator, the channel QFI of its corresponding channel $\mathcal{U}_{\theta\tilde{G}_{\alpha_c}}$ is proportional to the norm of the generator (see eq.(31)), giving

$$4\|\tilde{G}_{\alpha_c}\|_{\infty}^2 = C(\mathcal{U}_{\theta\tilde{G}_{\alpha_c}}; \theta) .$$

We have already shown that the channel QFI of a unitary channel is invariant under a θ -linear shift, eq.(28) Thus we have

$$C(\mathcal{U}_{\theta\tilde{G}_{\alpha_c}}; \theta) = C(\mathcal{U}_{\theta G}; \theta) .$$

Eventually, we have shown that

$$C(\mathcal{U}_{G_{\text{ext}}}; \theta) \leq C(\mathcal{U}_{\theta G}; \theta) . \tag{53}$$

□

IV. DISCUSSION AND CONCLUSION

Most of the work in quantum-enhanced measurements has focused on using entanglement in order to improve the scaling of the sensitivity with the number of probes. An alternative to the experimentally problematic multi-partite entanglement of a large number of probes is to use more general Hamiltonians, in particular Hamiltonians allowing for interactions between the subsystems, an approach known as “non-linear schemes” [10–12]. It was realized that the parameter characterizing a k -body interaction strength can be estimated with an uncertainty (measured by the standard deviation) that scales as $1/N^{k-1/2}$ for an initial product state of all N probes, and $1/N^k$ if the initial state is entangled. Similarly, “coherent averaging” was introduced and examined in detail in [14], based on earlier work on “decoherence-enhanced measurements” [13]. In both cases, the Hamiltonian has the structure typical of open quantum systems, $H = H_{\text{sys}} + H_{\text{int}} + H_{\text{env}}$, where H_{sys} corresponds to the N non-interacting subsystems introduced above, H_{env} describes an environment for the decoherence-enhanced measurements or a “quantum bus” for coherent averaging. Also there it was found that in a certain parameter regime interaction strength can be measured with Heisenberg-limited scaling *i.e.* a standard deviation scaling as $1/N$ — when measuring the quantum bus and using an initial product state. However, Heisenberg-limited scaling of the uncertainty of the original parameter θ coded in H_{sys} could only be achieved when

measuring the whole system, *i.e.* system plus quantum bus.

The results of the present work allow us to make strong statements for the quantum enhancements possible in such schemes based on more general Hamiltonians: First, we considered QFI itself rather than its scaling with N ; and secondly, we obtained bounds for the QFI corresponding to the original parameter to be estimated rather than for new parameters that characterize the interaction strength to the ancilla system introduced. Our theorem shows that the uncertainty of the estimation of the original parameter of a unitary phase shift channel cannot be reduced by an arbitrary Hamiltonian extension to a larger system. This implies in particular for the non-linear schemes that the $1/N$ scaling of the standard deviation of the estimate of the original parameter of the phase shift channel (*i.e.* the HL obtained when using a highly entangled state of all probes) cannot be improved upon by introducing k -body interactions. Also for coherent averaging or decoherence-enhanced measurements one cannot beat the HL scaling of the estimation of the level spacing of the probes that one can achieve at least theoretically by using a maximally entangled state of the N probes and no ancilla. Nevertheless, both non-linear schemes and coherent averaging still do have their interest: Sometimes it is important to know the precision with which an interaction can be measured (e.g. the gravitational constant [27]), and it is interesting that interactions can be measured more precisely than a phase shift for a large number of probes. Similarly, for coherent averaging, it is important that in certain parameter regimes HL scaling of the uncertainty of the original parameter (that characterizes e.g. level spacings of the probes) can be achieved with an initial product state of the probes, whereas HL scaling without the coherent averaging procedure requires a highly entangled initial state.

Our results were obtained for the estimation of a phase shift obtained from a generator with bounded spectrum. For more complex dependences of the Hamiltonian on the parameter to be estimated, the question is still open. A simple generalization is possible, however, when the Hamiltonian $H(\theta)$ and its derivative $dH(\theta)/d\theta = \dot{H}(\theta)$ commute: $[\dot{H}(\theta), H(\theta)] = 0$. Then theorem 2 is directly generalized by replacing G with $\dot{H}(\theta)$.

For generators with unbounded spectrum (e.g. the generator of a phase shift in one arm of a Mach-Zehnder interferometer, which is simply the photon number in that mode), the maximum variances (20) and (21) are formally infinite. Our theorem is still useful in such a context if we introduce a cut-off \hat{g} in the spectrum. If \hat{g} remains finite, *i.e.* $\hat{g} \in [g_{\min}, \infty[$, we

are left with a bounded operator and then our theorem applies. The cut-off \hat{g} can be made arbitrarily large, which is enough for typical physical applications of quantum metrology. Finding the maximal possible QFI is not the end of the road either: One would like to know the optimal state, and also the optimal POVM (which we do not discuss here). Another question that we have left open is whether the bound derived here is always reachable.

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